ON THE PROBABILITY DISTRIBUTION OF THE DIFFERENCE OF VELOCITIES AT TWO DISTINCT POINTS OF A HOMOGENEOUS, ISOTROPIC TURBULENT FLOW

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We consider an equation defining a characteristic function of the probability that the velocities at two distinct points of a homogeneous, isotropic turbulent flow, differ from each other. This equation can be derived from the hydrodynamic equations for an incompressible fluid under two assumptions, and has the form

$$\frac{\partial f}{\partial t} = i\frac{\partial^2 f}{\partial \lambda_k \partial r_k} + \frac{2}{3}\varepsilon \lambda_k^2 f$$

where t denotes time, $r_j = x_j^{(1)} - x_j^{(2)}$ ($j = 1, 2, 3$), $x_j^{(s)}$ ($s = 1, 2$) denote the coordinates of the two points in question, $\varepsilon = -\frac{1}{2} d\langle u^2_j\rangle/dt$ is the rate of dissipation of kinetic energy of the flow, $u_j(x^{(s)})$ is the velocity, $f(\lambda) = \int \exp(i\lambda_j v_j) P(v) d^3 v$, $P(\mathbf{v})$ is the probability distribution of the difference of velocities and $v_j = u_j(\mathbf{x}^{(1)}) - u_j(\mathbf{x}^{(2)})$. We shall introduce the following magnitudes

$$\varphi = \exp(i\lambda_j v_j), \qquad \varepsilon_{kj}^{(s)} = \frac{\partial u_k(\mathbf{x}^{(s)})}{\partial x_l^{(s)}} \frac{\partial u_j(\mathbf{x}^{(s)})}{\partial x_l^{(s)}}$$

Mean values of φ is the required characteristic function and the mean value of $\varepsilon_{kj}^{(s)}$ is proportional to the dissipation of the kinetic energy of the flow.

We shall assume that φ and $\varepsilon_{kj}^{(s)}$ are statistically independent provided that the Reynolds' number is large and the distance between the two points is large compared with $\eta = \nu^{3/4}\varepsilon^{-1/4}$. Here ν is the coefficient of kinematic viscosity. The magnitude $\varepsilon_{kj}^{(s)}$ can be defined in terms of Fourier coefficients $dZ_k(\varkappa')$ where the wave numbers $\varkappa' \sim 1/\eta$. Here

$$dZ_k(\varkappa) = \frac{1}{(2\pi)^3} \times$$

$$\times \int u_k(\mathbf{x}) e^{-i\varkappa\cdot\mathbf{x}} \left[\frac{\exp(-ix d\varkappa_1) - 1}{-ix_1}\right]\left[\frac{\exp(-ix_2 d\varkappa_2) - 1}{-ix_2}\right]\left[\frac{\exp(-ix_3 d\varkappa_3) - 1}{-ix_3}\right] d^3x$$

The magnitude φ can be defined in terms of Fourier coefficients $dZ_k(\varkappa'')$ where the wave numbers $\varkappa'' \sim 1/|\mathbf{x}^{(1)} - \mathbf{x}^{(2)}| = 1/r$. If the Reynolds' number is large and $r \gg \eta$, then $dZ_k(\varkappa')$ and $dZ_k(\varkappa'')$ are, according to Kolmogorov [1], statistically independent. In view of this, our previous assertion is sufficiently probable.

We shall also assume that there exists a correlation between φ and ψ_k. Here

$$\psi_k = \frac{\partial p(\mathbf{x}^{(1)})}{\partial x_k^{(1)}} - \frac{\partial p(\mathbf{x}^{(2)})}{\partial x_k^{(2)}}$$

and p is the pressure divided by the density.

Mean value of the product of the above magnitudes characterizes the effect of pressure forces on turbulence. It is generally accepted that this effect reduces to redistribution of the energy between two vortices of the same size but different spatial orientation and results in the tendency of the turbulence to become isotropic. As soon as it happens we can in the first approximation, neglect the above effect.

Under those assumptions, the equations of hydrodynamics easily yield the required equations. We have

$$\frac{\partial \langle\varphi\rangle}{\partial t} = i\lambda_k \left\langle \varphi \left(\frac{\partial u_k(\mathbf{x}^{(1)})}{\partial t} - \frac{\partial u_k(\mathbf{x}^{(2)})}{\partial t} \right) \right\rangle = i\lambda_k \left\langle \left(- u_j(\mathbf{x}^{(1)}) \frac{\partial u_k(\mathbf{x}^{(1)})}{\partial x_j^{(1)}} + \right.\right.$$

$$\left.\left. + u_j(\mathbf{x}^{(2)}) \frac{\partial u_k(\mathbf{x}^{(2)})}{\partial x_j^{(2)}} - \psi_k + \nu\Delta u_k(\mathbf{x}^{(1)}) - \nu\Delta u_k(\mathbf{x}^{(2)}) \right) \right\rangle \quad (1)$$

$$i\lambda_k \left\langle \varphi \left(- u_j(\mathbf{x}^{(1)}) \frac{\partial u_k(\mathbf{x}^{(1)})}{\partial x_j^{(1)}} + u_j(\mathbf{x}^{(2)}) \frac{\partial u_k(\mathbf{x}^{(2)})}{\partial x_j^{(2)}} \right) \right\rangle =$$

$$= - \frac{\partial}{\partial x_k^{(1)}} \langle u_k(\mathbf{x}^{(1)})\varphi\rangle - \frac{\partial}{\partial x_k^{(2)}} \langle u_k(\mathbf{x}^{(2)})\varphi\rangle = - \frac{\partial}{\partial r_k} \langle \varphi v_k\rangle = i \frac{\partial^2 \langle\varphi\rangle}{\partial\lambda_k \partial r_k} \quad (2)$$

Our first assumption leads to

$$2\nu \frac{\partial^2 \langle\varphi\rangle}{\partial r_k^2} = i\nu\lambda_k \langle \varphi (\Delta u_k(\mathbf{x}^{(1)}) - \Delta u_k(\mathbf{x}^{(2)}))\rangle - 2\nu\lambda_j\lambda_k \langle\varepsilon_{jk}\rangle \langle\varphi\rangle \quad (3)$$

while the second assumption yields

$$i\lambda_k \langle\varphi\psi_k\rangle = 0. \quad (4)$$

Formulas (1) to (4) give

$$\frac{\partial f}{\partial t} = i \frac{\partial^2 f}{\partial\lambda_k \partial r_k} + 2\nu \frac{\partial^2 f}{\partial r_k^2} + 2\nu \langle\varepsilon_{jk}\rangle \lambda_j\lambda_k f$$

When $r \gg \eta$, $2\nu\partial^2 f/\partial r_k^2$ is vanishingly small and

$$\frac{\partial f}{\partial t} = i \frac{\partial^2 f}{\partial\lambda_k \partial r_k} + \frac{2}{3} \varepsilon\lambda_k^2 f \quad (5)$$

In the derivation of (5) we have used the relation $\nu\langle\varepsilon_{jk}\rangle = 1/3\, \varepsilon\delta_{jk}$ which follows from the isotropic character and homogeneity of the flow.

Eq. (5) should be solved with the following boundary conditions: $f = 1$ when $r = 0$ and $\partial f/\partial\lambda_k = 0$ when $|\lambda| = 0$. The last condition follows from the relation $\langle v_k\rangle = 0$.

We can integrate (5) in quadratures. Before doing it, let us consider its properties and two limit cases which it describes.

Expanding f into the Taylor series in λ, inserting it into (5) and equating the coefficient accompanying λ_k to zero, we obtain a continuity equation for the second moment of the velocity field. Equating to zero the coefficient of $\lambda_k\lambda_j$ yields the well known von Kármánn-Howarth equation. Later we shall derive relations connecting the moments of orders n, $n + 1$ and $n - 2$ (where n is an integer).

Applying the inverse Fourier transform to (5) we obtain an expression for the probability distribution of the difference of velocities at two distinct points in the form

$$\frac{\partial P}{\partial t} = - v_k \frac{\partial P}{\partial r_k} - \frac{2}{3} \frac{\partial^2 P}{\partial v_k^2}$$

resembling a diffusion equation of an inert component in the six-dimensional (v_j, r_j)-space.

Let us consider the behavior of (5) as $r \to \infty$, assuming that $\langle u_k\rangle = 0$. All point moments of odd orders are, by the virtue of isotropy, equal to zero and we have, in this case, $f \to F^2(\lambda)$, where F is the characteristic probability function of velocity at a single point. Consequently, when $r \to \infty$, (5) yields

$$\frac{\partial F}{\partial t} = \frac{1}{3}\varepsilon\lambda_k^2 F \tag{6}$$

from which we find

$$F(\lambda, t) = F(\lambda, 0)\ \exp\left[-{}^1/_6\ (\langle u_k^2\rangle - \langle u_k^2(0)\rangle)\lambda_k^2\right]$$

If $\langle u_k^2\rangle = 0$, then $F = 1$ and

$$F(\lambda, 0) = \exp\left[-{}^1/_6\,\lambda_k^2\langle u_k^2(0)\rangle\right] \tag{7}$$

Therefore

$$F = \exp\left[-{}^1/_6\,\lambda_k^2\,\langle u_k^2\rangle\right]$$

i.e. the probability distribution of velocity at a point obeys the usual law and this confirms the experimental data of Townsend [2].

Formula (7) shows that Eq. (6) need not have a physically plausible solution with any type of initial conditions. This may come as a surprise until we remember that Eq. (6) was obtained under definite assumptions which were valid when the turbulence was in a - so called - "equilibrium" state. A certain amount of time elapses before this state is reached and the function F assumes the form (7) during this period.

Let us consider another limit case when r is small enough to fall within the inertial interval. In addition, from the dimensional analysis and isotropy we can infer that f should depend on the following two variables only

$$z = \sqrt{\lambda_k^2}(\varepsilon r)^{1/3}, \qquad y = \lambda_j r_j / r\,\sqrt{\lambda_k^2}$$

Taking this into account we obtain, from (5),

$$\frac{1}{3} z\frac{\partial f}{\partial z} r^{2/3}\varepsilon^{-4/3}\frac{d\varepsilon}{dt} = i\left[\frac{1}{3} y\frac{\partial f}{\partial z} + \frac{1}{3} yz\frac{\partial^2 f}{\partial z^2} + \frac{4}{3}(1-y^2)\frac{\partial^2 f}{\partial z\,\partial y} - \right.$$
$$\left. - \frac{1}{z} y(1-y^2)\frac{\partial^2 f}{\partial y^2} + \frac{1}{z}(1+y^2)\frac{\partial f}{\partial y}\right] + \frac{2}{3}z^2 f$$

which in turn yields

$$\frac{1}{3} y\frac{\partial}{\partial z}\left(z\frac{\partial f}{\partial z}\right) + \frac{4}{3}(1-y^2)\frac{\partial^2 f}{\partial y\,\partial z} - \frac{1}{z} y(1-y^2)\frac{\partial^2 f}{\partial y^2} + \frac{1}{z}(1+y^2)\frac{\partial f}{\partial y} - \frac{2}{3} iz^2 f = 0 \tag{8}$$

when $r \to 0$ or when $t \to \infty$.

The solution of (8) with the conditions that $f = 1$ and $\partial f/\partial z = 0$ when $r = 0$, represents the characteristic function of the required probability distribution in the inertial interval.

Eq. (8) is hyperbolic for all values of z and y, which are physically meaningful ($0 \leqslant z < < \infty$, $|y| \leqslant 1$). Since the line $z = 0$ represents its characteristic, we have two possibilities: (1) – a solution does not exist; (2) – a solution exists and is not unique. Simple analysis shows that the condition of compatibility holds, hence the assertion (2) is valid.

Eq. (8) connects the moments of orders n and $n + 3$. The relation is constructed in such a manner, that the $(2n + 3)$-th order moment is expressed in terms of the $2n$-th order moment uniquely, while the $2n$-th order moment is given in terms of the $(2n - 3)$-th order moment and includes, in addition, an arbitrary constant.

Indeed, let us write f as

$$f = \sum_{n=0}^{\infty}\sum_{m=0}^{n} A_{n,m}\frac{i^n z^n y^m}{n!}$$

(where m is an integer) and let us insert it into (8). Equating to zero the coefficients of $z^{n-1}y^{m-1}$ gives

$$A_{n,m} = -\frac{(n-m+2)(n-3m+6)}{m(4n-3m+6)} A_{n,m-2} - \frac{2n(n-1)(n-2)}{m(4n-3m+6)} A_{n-3,m+1}$$

from which we easily see that when n is odd ($m = 1, 3, \ldots, n$) then we have the same number of equations and unknowns, while for the even n ($m = 0, 2, \ldots, n$) the number of unknowns exceeds the number of equations by one.

As an example, we shall give first six moments different from zero

$$\langle V_n^2\rangle = A_{2,0}, \quad \langle V_p^2\rangle = {}^3/_4\, A_{2,0}$$
$$\langle V_n^2 V_p\rangle = -{}^4/_{15}, \quad \langle V_p^3\rangle = -{}^4/_5$$
$$\langle V_n^4\rangle = A_{4,0}, \quad \langle V_n^2 V_p^2\rangle = {}^1/_4\, A_{4,0}; \quad \langle V_p^4\rangle = {}^9/_{20}\, A_{4,0}$$
$$\langle V_n^4 V_p\rangle = -{}^{24}/_{23}\, A_{2,0}, \quad \langle V_n^2 V_p^3\rangle = -{}^{353}/_{391}\, A_{2,0}, \quad \langle V_p^5\rangle = -{}^{1410}/_{391}\, A_{2,0} \tag{9}$$
$$\langle V_n^6\rangle = A_{6,0}, \quad \langle V_n^4 V_p^2\rangle = {}^3/_{20}\, A_{6,0} + {}^4/_{15}, \quad \langle V_n^2 V_p^4\rangle = {}^1/_{10}\, A_{6.0} + {}^8/_{15} \quad \langle V_p^6\rangle = {}^1/_4\, A_{6,0} + 4$$
$$\langle V_n^7\rangle = -{}^{60}/_{31}\, A_{4,0}; \quad \langle V_n^4 V_p^3\rangle = -{}^{5294}/_{5425}\, A_{4,0}; \langle V_n^2 V_p^5\rangle = -{}^{33}/_{25}\, A_{4,0} \langle V_p^7\rangle = -{}^{4221}/_{775}\, A_{4,0}$$

Here p refers to the component of the vector $\mathbf{V}$ parallel to $\mathbf{r}$, n – to the component perpendicular to $\mathbf{r}$ and $\mathbf{V} = \mathbf{v}(\varepsilon r)^{-1/3}$. Formulas (9) yield

$$\langle v_p^5\rangle / (\langle v_p^2\rangle \langle v_p^3\rangle) = {}^{2350}/_{391}, \qquad \langle v_p^3 v_n^2\rangle / (\langle v_p^3\rangle \langle v_n^2\rangle) = {}^{1765}/_{1564}$$
$$\langle v_p^7\rangle / (\langle v_p^4\rangle \langle v_p^3\rangle) = {}^{469}/_{31}, \qquad \langle v_p^3 v_n^4\rangle / (\langle v_p^3\rangle \langle v_n^4\rangle) = {}^{2647}/_{1805}$$

Constants $A_{4,0}$ and $A_{6,0}$ cannot be arbitrary. Inequalities

$$(\langle v_p^3\rangle)^2 \leqslant \langle v_p^2\rangle\, [\langle v_p^4\rangle - (\langle v_p^2\rangle)^2], \qquad \langle v_n^6\rangle \geqslant (\langle v_n^4\rangle)^{3/2}$$

together with (9), yield

$$A_{4,0} \geqslant {}^{20}/_3\, ({}^8/_{25})^{2/3}, \quad A_{6,0} \geqslant {}^8/_{25}\, ({}^{20}/_3)^{3/2}$$

Study of higher order moments leads us to the conclusion that the magnitudes $A_{2n,0}$ are always restricted in a certain way, namely, we require that each $A_{2n,0}$ should be not smaller than a certain number depending on $A_{2n-2,0}, A_{2n-4,0}, \ldots, A_{2,0}$.

The above constants cannot have very large values, because the series defining the characteristic function will then not converge. This allows us to obtain a definite upper bound for the values of $A_{2n,0}$ using the equation for the probability distribution of velocity differences, which has the form

$$-w\alpha\Pi - \frac{1}{3}w^2\alpha\frac{\partial\Pi}{\partial w} + w(1-\alpha^2)\frac{\partial\Pi}{\partial\alpha} + \frac{2}{3w^2}\frac{\partial}{\partial w}\left(w^2\frac{\partial\Pi}{\partial w}\right) + \frac{2}{3w^2}\frac{\partial}{\partial\alpha}\left[(1-\alpha^2)\frac{\partial\Pi}{\partial\alpha}\right] = 0$$
$$\left(w = \sqrt{v_k^2}\,(\varepsilon r)^{-1/3}, \quad \alpha = v_j r_j / r\sqrt{v_k^2}, \ \Pi = P\,(\varepsilon r)^{-1}\right)$$

within the inertial interval and is elliptic. We may find that its solution need not be unique since the coefficients accompanying the unknown function $(-w\alpha)$ changes its sign. However in this case another restriction comes into force namely: that the function $\Pi(w)$ should diminish sufficiently fast for the integrals of the form

$$\int_0^\infty \int_{-1}^1 \Pi\,(w;\,\alpha)\, w^n\, d\alpha\, dw$$

to converge.

Let us now obtain the general solution of (5). The function $f - F^2$ obviously satisfies (5), and its Fourier transform exists. Therefore we can apply Fourier transformation to (5), in which f is replaced by $f - F^2$. This results in

$$\frac{\partial\Phi}{\partial t} = \varkappa_k \frac{\partial\Phi}{\partial\lambda_k} + \frac{2}{3}\varepsilon\lambda_k^2\Phi$$

where Φ is the Fourier transform of $f - F^2$. This in turn yields

$$f = F^2 + \int \Phi\,[\sqrt{\varkappa_k^2},\ \lambda_k^2(1-\beta^2),\ \sqrt{\varkappa_k^2}t + \lambda\beta] \times$$
$$\times \exp\left[{}^2/_3\,\lambda_k^2(1-\beta^2)\int_0^t \varepsilon(s)\,ds - {}^2/_3\int_0^t (\sqrt{\varkappa_k^2}\,t + \lambda\beta - \sqrt{\varkappa_k^2}\,s)^2 \times\right.$$
$$\left.\times\, \varepsilon(s)\,ds - i\sqrt{\varkappa_k^2}\,r\gamma\right]\varkappa_k^2\,(1 - \beta^2 - \gamma^2 - y^2 + 2\beta\gamma y)^{-1/2}\, d\,(\sqrt{\varkappa_k^2})\, d\beta\, d\gamma$$
$$(\beta = \lambda_j\varkappa_j / \sqrt{\lambda_k^2\varkappa_k^2}, \quad \gamma = r_j\varkappa_j / r\sqrt{\varkappa_k^2})$$

which does not however contribute much information, since as usual, the initial conditions are not known and cannot be set arbitrarily.

The above ideas can also be applied to determining the probability distribution of the concentration of an inert component which is statistically homogeneous, in a homogeneous turbulent flow. It is sufficient to assume that the magnitudes $e^{i\mu c}$ and $D\partial^2 c/\partial x_k^2 = \chi$ are statistically independent provided that the Reynolds' and Peclet numbers are sufficiently large. Here c is the concentration and D is the coefficient of molecular diffusion. Analogous procedure yields

$$\partial \langle e^{i\mu c}\rangle / \partial t = \langle\chi\rangle \mu^2 \langle e^{i\mu c}\rangle \qquad (10)$$

which resembles (6) and where $\langle\chi\rangle$ characterising the rate of mixing of the substance down to the molecular level, is analogous to ε.

Solution of (10) with the condition that $\langle e^{i\mu c}\rangle = e^{i\mu\langle c\rangle}$ when $\langle c^2\rangle = \langle c\rangle^2$, has the form

$$\langle e^{i\mu c}\rangle = \exp\left[-\tfrac{1}{2}\left(\langle c^2\rangle - \langle c\rangle^2\right)\mu^2 + i\mu\langle c\rangle\right]$$

i.e. the probability distribution of the concentration of the added substance obeys the usual law.

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